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Free vibration analysis of circular plates with variable thickness by the generalized differential quadrature rule

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Abstract

The generalized differential quadrature rule proposed recently by these authors is applied here to the free vibration analyses of solid circular plates with radially varying thickness and elastic restraints. The thickness can vary radially in any given continuous form, and it varies exponentially and linearly in this work. Two regularity conditions corresponding to the plate center are expressed in explicit formulae, since they have been inexactly or even wrongly expressed in the literature. Such errors are pointed out in the application of the differential quadrature techniques and in the expression of the regularity conditions at the plate center. Numerical results are presented for a number of plates, illustrating the versatility and accuracy of the approach. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Circular plate; Variable thickness; Ring supports; Elastic restraint; Free vibration; Differential quadrature method; Collocation method; Pseudospectral method; Spectral method

1. Introduction

Circular plates have many applications in civil and mechanical engineering. In reality, several complicating factors may come into play: nonuniform thickness, elastic constraints, intermediate supports, and anisotropic, composite, or laminated materials. Researchers have used various methods of analyses for such plates. The book of Leissa (1969) is an excellent source of available information. Since then, he has published many other survey articles on the subject. Thus, an attempt to provide a complete review here is superfluous. This paper deals with the free vibration of solid circular plates with radially variable thickness and elastic constraints, using the generalized differential quadrature rule (GDQR) proposed recently by Wu and Liu (1999a,b, 2000a,b, 2001a). The thickness of the circular plates can vary radially in any given continuous form, such as in exponential and linear form in this work. Examples are presented and excellent results have been obtained.

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Bellman and Casti (1971) first proposed the differential quadrature method (DQM). Civan and Sliepcevich (1983, 1984) were first to extend the DQM to engineering problems. The conventional DQM is usually applied to differential equations of not more than second order (Bellomo, 1997; Bert and Malik, 1996). Striz et al. (1988) first solved circular plate problems using the DQM. Jang et al. (1989) extended the DQM to high order, for example fourth order, boundary value differential equations in solid mechanics. Striz et al. (1997) and Chen et al. (2000) used the quadrature element method, which can be categorised as a high-order finite element method. The GDQR is a generalization of the DQM (Wu and Liu, 2000a) to any high-order differential equation in a strict form. The GDQR has been applied to initial value differential equations of second- to fourth- orders for the first time by Wu and Liu (1999b, 2000b) and Liu and Wu (2000) and to two-point boundary value differential equations of fourth, sixth, and eighth orders in solid mechanics by Wu and Liu (1999a,b, 2000a,c,d, 2001a,b) and Wu and Liu (2001a,b) and in fluid mechanics by Liu and Wu (2001c). The GDQR is first simultaneously applied to the spatial and temporal dimensions for initial-boundary-value problems by Liu and Wu (2001d). Multipoint ($\geqslant 3$ points) boundary value problems are first solved by Wu and Liu (2001c) using the DQM. Wu and Liu (2001a) have completed static and free vibrational analyses of two-dimensional rectangular plates using the GDQR, while Wang et al. (1998) and Chen et al. (1997) coped with the identical problems. Wu and Liu (2001a) applied three boundary conditions at any corner of rectangular plates, while Wang et al. (1998) and Chen et al. (1997) used four boundary conditions at any corner of rectangular plates. A close examination suggests that there can only be three boundary conditions at any corner of rectangular plates (Wu and Liu, 2001a).

The number of regularity conditions at a solid circular plate center is another very interesting issue. Note that the conditions at a solid circular plate center are called regularity conditions. The governing equations of free vibration of circular plates can be expressed as fourth-order ordinary differential equations, if the number of circumferential nodal lines is assumed under symmetrically applied uniform boundary conditions. It is common knowledge that a fourth-order differential equation should have four given conditions to determine the four integration coefficients in the general expression of its analytic solution. One important thing should be kept in mind that the DQ techniques implement a differential equation and its given conditions directly in a strong form. However, Gu and Wang (1997) and Gutierrez et al. (1996) employed only three (two for the outer edge and one for the center) conditions for solid circular plates and obtained results using the DQ techniques. It is unimaginable that one would apply the DQM to Euler beam problems using only three boundary conditions. This work discusses in detail the use of four conditions for the free vibration of solid circular plates. The two regularity conditions at a solid circular plate center are explicitly expressed in this work.

Four boundary conditions were applied to the free vibration analysis of annular plates using the DQM (Wang et al., 1995; Romanelli et al., 1998), while only three conditions were employed for the free vibration analysis of solid circular plates as pointed out just before. Romanelli et al. (1998) studied a circular uniform annular plate with an intermediate circular support and a free inner edge. The fundamental frequencies obtained using the DQM differed by approximately 10% from the values obtained by means of the Raleigh–Ritz and finite element methods. For some cases, the DQM did not provide satisfactory accuracy as stated by Romanelli et al. (1998). However, the reason why those bad results occurred was not mentioned. It is apparent that an error must have occurred in the application of the DQM. Domain decomposition should have been employed at the intermediate circular support, because a shear force discontinuity exists there, and trial functions with infinite order derivatives are employed in the DQM. Domain decomposition has been used and very accurate results have been obtained in the application of the DQ techniques to multispan Euler beams and stepped cross-section problems by Wu and Liu (2001a,b) and Wu and Liu (2000c).

This work deals with the free vibration of solid circular plates with radially variable thickness and elastic constraints using the GDQR. Two regularity conditions at the solid circular plate origin are discussed in detail. Numerical results are presented for a number of examples, illustrating the versatility and accuracy of the approach.

2. Regularity conditions at the center

A detailed study of the regularity conditions at the solid circular plate center has been carried out in many good books (Reddy, 1999; Gould, 1999; Weaver et al., 1990; Szilard, 1974; Xu, 1982). Reddy (1999) pointed out the two conditions at the center for axisymmetrically loaded circular plates in the static analysis. They are zero rotation angle and shear force expression. For free vibration problems, the following contents are found in almost all books and other literature.

When a circular plate of isotropic material possesses uniform thickness, the governing equation for the free transverse vibration of the plate is expressed as

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 W+\frac{\rho t\omega^2}{D}W=0, \tag{1}$$

where r and θ are polar coordinates. W is the transverse displacement, ρ the density of the material, t the plate thickness, ω the circular natural frequency, and D the flexural rigidity, $D = Et^3/[12(1-\nu^2)]$. E is the elasticity coefficient and ν is Poisson's ratio.

If uniform boundary conditions are symmetrically applied about a diameter of the plate, the solution can be assumed in the form of the Fourier series

$$W(r,\theta)=\sum_{k=1}^{\infty} w_{(k)}(r)\cos k\theta. \tag{2}$$

Its normal modes are described in a classical fashion using Bessel functions

$$W(r,\theta)=\sum_{k=0}^{\infty}[C_{1k}J_k(\lambda r)+C_{2k}Y_k(\lambda r)+C_{3k}I_k(\lambda r)+C_{4k}K_k(\lambda r)]\cos k\theta, \tag{3}$$

where $\lambda^4=\rho t\omega^2/D$. J_k and Y_k are the Bessel functions of first and second kind, respectively, and I_k and K_k are the modified Bessel functions of first and second kind, respectively. k is the number of circumferential nodal lines. C_{1k}, C_{2k}, C_{3k}, and C_{4k} are integration constants.

For solid circular plates, the terms involving Y_k and K_k in Eq. (3) must be discarded in order to avoid a singularity of the deflections and stresses (i.e., avoid infinite values) at the plate origin, $r=0$. Then, the kth term of Eq. (3) becomes

$$W_k(r,\theta)=[C_{1k}J_k(\lambda r)+C_{3k}I_k(\lambda r)]\cos k\theta. \tag{4}$$

Next, two homogeneous boundary conditions at the outer edge are applied to obtain the natural frequencies for any number of the circumferential nodal line k.

For a fourth-order differential equation with four integration coefficients in the general analytic solution, four conditions must be used. This law is surely suitable for Eqs. (1) and (3). However, the conditions at the center of a circular plate are not mentioned at all due to the special characteristics of the Bessel functions. Gu and Wang (1997) used the condition at the center in a strong form and applied only one condition at the center as follows

$$\mathrm{d}w_{(k)}/\mathrm{d}r+K_0 w_{(k)}=0, \tag{5}$$

where the constant K_0 is set to zero when $k=0$ and to ∞ when $k>0$. This condition means zero rotation angle when $k=0$ or zero displacement when $k>0$. The correct expression is that the mode is symmetric and the rotation angle at the center is zero for any even number of $k=0,2,4,\ldots$, and that the mode is antisymmetric and the displacement at the center is zero for any odd number of $k=1,3,5,\ldots$ Gutierrez et al. (1996) also employed only one condition at the origin and obtained the results for $k=0$ only.

Many papers used the conventional Ritz method to obtain the natural frequency of solid circular plates (Avalos et al., 1988; Laura et al., 1987; Gutierrez et al., 1996). A suitable approximation for the displacement function should satisfy the essential boundary conditions in the Ritz method. However, the following displacement function has been assumed as (Avalos et al., 1988)

$$W_k(r,\theta) = \cos k\theta \sum_{j=0}^{J} A_{jk} \left[c_{jk}\left(\frac{r}{a}\right)^{\gamma} + d_{jk}\left(\frac{r}{a}\right)^{2} + 1 \right] \left(\frac{r}{a}\right)^{j+k}, \tag{6}$$

where a is the radius of the circular plate and γ an optimum variable. c_{jk} and d_{jk} are constants determined by satisfying the boundary conditions at the outer edge.

The displacement at the origin in Eq. (6) is always zero if $j + k \geqslant 1$. This is true for all $k \geqslant 1$, as supposed in Eq. (5) by Gu and Wang (1997). It is apparent that both displacement and rotation angle are zero at the origin for items with $j + k \geqslant 2$. This means that the center is clamped. This function approximation is obviously an improper presentation of the actual situation. Therefore, it is not surprising that the higher order frequencies obtained there have very large errors.

Takabatake et al. (1996) studied the free vibration of a circular plate with voids by means of the Galerkin method. The following shape functions were used for a simply supported plate and clamped plate, respectively:

$$W_k(r,\theta) = \cos k\theta \sum_{m} A_{km} \sin\left[\frac{m\pi}{2}\left(1 - \frac{r}{a}\right)\right], \tag{7a}$$

$$W_k(r,\theta) = \cos k\theta \sum_{m} A_{km} \sin\left[\frac{m\pi}{2}\left(1 - \frac{r}{a}\right)\right] \sin\left[\frac{\pi}{2}\left(1 - \frac{r}{a}\right)\right], \tag{7b}$$

in which m is the number of radial nodal lines, and m takes an odd number for $k = 0$ and an even number for $k \geqslant 1$. The essential conditions of Eqs. (7a) and (7b) coincide with Eq. (5) and, thus, have the same problem.

Kim and Dickinson (1989) considered many complicating effects of circular plates using the Rayleigh–Ritz method. A solid circular plate was simulated by permitting the inner radius of an annular plate to become very small and circumambulating the conditions at the center. Gutierrez et al. (1996) and Singh and Saxena (1995, 1996) even considered only the case of $k = 0$ since this case is similar to static analysis. It is well known that the lowest frequency may not correspond to $k = 0$ in some kinds of boundary conditions.

Gorman (1982) dealt with the free vibration analyses of rectangular plates, where symmetric and antisymmetric characteristics and the method of superposition were widely employed. If the number of circumferential nodal lines k is assumed, circular plates also have symmetric modes for even k and antisymmetric modes for odd k. Therefore, the authors write the regularity conditions at the solid circular plate center as follows:

$$w = 0, \quad M_r = 0, \quad (k = 1, 3, 5, 7 \ldots), \tag{8}$$

$$\mathrm{d}w/\mathrm{d}r = 0, \quad V_r = 0, \quad (k = 0, 2, 4, 6 \ldots), \tag{9}$$

where M_r is the radial bending moment and V_r the effective radial shear force.

Examples to be presented in Section 5 demonstrate that these two regularity conditions at the center of the solid circular plate give correct representations.

3. The generalized differential quadrature rule

The GDQR is briefly described here for the completeness of the present work. A field function $\psi(x)$, governed by a differential equation, is constrained by one or more than one conditions at any individual discrete point. The solution domain is divided into points $x_i (i = 1, 2, \ldots, N)$ that include all the points with given conditions. Note that only the governing equation is to be satisfied for some points. If n_i conditions (equations) are to be satisfied at point x_i, the GDQR is expressed as follows (Wu and Liu, 1999a,b, 2000a,b)

$$\frac{\mathrm{d}^r\psi(x_i)}{\mathrm{d}x^r} = \sum_{j=1}^{N}\sum_{l=0}^{n_j-1} E_{ijl}^{(r)}\psi_j^{(l)} = \sum_{j=1}^{M} E_{ij}^{(r)} U_j \quad (i = 1, 2, \ldots, N), \tag{10}$$

where $E_{ij}^{(r)}$ (which is a convenient expression of $E_{ijl}^{(r)}$) is the weighting coefficient corresponding to the rth-order derivative at point x_i, and $M = \sum_{i=1}^{N} n_i$ is the number of the total independent variables U_j, which is expressed in a series as

$$\{U\}^{\mathrm{T}} = \{U_1, U_2, \ldots, U_j \ldots, U_M\} = \left\{\psi_1^{(0)}, \psi_1^{(1)}, \ldots, \psi_1^{(n_1-1)}, \ldots, \psi_N^{(0)}, \psi_N^{(1)}, \ldots, \psi_N^{(n_N-1)}\right\}, \tag{11}$$

where $\psi_i = \psi_i^{(0)} = \psi(x_i)$ are the function values, and $\psi_i^{(l)} = \psi^{(l)}(x_i)$ $(l = 1, 2, \ldots, n_i - 1)$ are their lth-order derivatives.

It is clearly shown from Eq. (10) that the GDQR forces the same number of independent variables $\psi^{(l)}(x_i)$ $(l = 0, 1, 2, \ldots, n_i - 1)$ as that of the equations at a point, and that its independent variables are chosen as the function values and their derivatives of possible lowest order wherever necessary. The DQM only chooses function values as independent variables. When $n_i = 1 (i = 1, 2, \ldots, N)$ are used in Eq. (10), the GDQR becomes the DQM as

$$\frac{\mathrm{d}^r\psi(x_i)}{\mathrm{d}x^r} = \sum_{j=1}^{N} E_{ij}^{(r)}\psi_j \quad (i = 1, 2, \ldots N). \tag{12}$$

4. Application of the generalized differential quadrature rule

Using a dimensionless coordinate $R = r/a$, the thickness and flexural rigidity of a circular plate can be expressed as (Fig. 1)

$$t = t_0 f(R), \quad D = D_0 f^3(R) = D_0 g(R), \tag{13}$$

where t_0 is the thickness at the center and $D_0 = Et_0^3/[12(1 - \nu^2)]$.

Linear and exponential thickness variations are used as examples

$$t = t_0(1 + \eta R), \quad t = t_0 \mathrm{e}^{\zeta R}, \tag{14}$$

where constants η and ζ can be positive or negative.

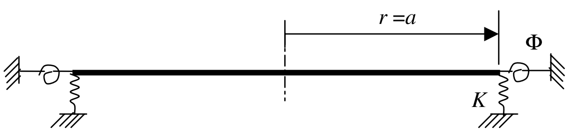

Fig. 1. A circular plate with variable thickness $t = t_0 f(R)$.

The governing partial differential equation and resultant forces of circular plates with radially varying thickness have been expressed by Reddy (1999) and Takabatake et al. (1996). For convenience, the subscript k of $w_{(k)}$ is dropped in the following expressions. Substituting Eq. (2) into the governing partial differential equation and using the dimensionless coordinate $R = r/a$ produce

$$g(R)\left(\frac{d^4w}{dR^4}+\frac{2}{R}\frac{d^3w}{dR^3}-\frac{1+2k^2}{R^2}\frac{d^2w}{dR^2}+\frac{1+2k^2}{R^3}\frac{dw}{dR}+\frac{(k^4-4k^2)}{R^4}w\right)+\frac{dg(R)}{dR}\left(2\frac{d^3w}{dR^3}+\frac{2+v}{R}\frac{d^2w}{dR^2}-\frac{1+2k^2}{R^2}\frac{dw}{dR}+\frac{3k^2}{R^3}w\right)+\frac{d^2g(R)}{dR^2}\left[\frac{d^2w}{dR^2}+v\left(\frac{1}{R}\frac{dw}{dR}-\frac{k^2}{R^2}w\right)\right]=\frac{\rho t_0 a^4\omega^2 f(R)}{D_0}w. \tag{15}$$

The radial bending moment M_r and effective shear force V_r are written for any k as

$$M_r = -\frac{D_0 g(R)}{a^2}\left[\frac{d^2w}{dR^2}+v\left(\frac{1}{R}\frac{dw}{dR}-\frac{k^2}{R^2}w\right)\right], \tag{16}$$

$$V_r = -\frac{D_0 g(R)}{a^3}\left(\frac{d^3w}{dR^3}+\frac{1}{R}\frac{d^2w}{dR^2}+\frac{vk^2-2k^2-1}{R^2}\frac{dw}{dR}+\frac{3k^2-vk^2}{R^3}w\right)-\frac{D_0}{a^3}\frac{dg(R)}{dR}\left(\frac{d^2w}{dR^2}+\frac{v}{R}\frac{dw}{dR}-\frac{vk^2}{R^2}w\right). \tag{17}$$

Using Eqs. (16) and (17), Eqs. (8) (second term) and (9) (second term) are re-written in term of w. However, the $R = 0$s are in the denominators of Eqs. (16) and (17). Note that the Y_k and K_k terms in Eq. (3) have been discarded to avoid a singularity. The terms with $R = 0$ in the denominators can, thus, be eliminated for the same reason. Therefore, Eqs. (8) and (9) are simplified as

$$w|_{R=0} = 0, \quad \left.\frac{d^2w}{dR^2}\right|_{R=0} = 0, \qquad (k = 1, 3, 5, 7, \ldots) \tag{18}$$

$$\left.\frac{dw}{dR}\right|_{R=0} = 0, \quad \left.\left(\frac{d^3w}{dR^3}+\frac{1}{g(R)}\frac{dg(R)}{dR}\frac{d^2w}{dR^2}\right)\right|_{R=0} = 0, \quad (k = 0, 2, 4, 6, \ldots). \tag{19}$$

The boundary conditions at the outer edge ($R = 1$) are expressed for elastic supports as

$$g(1)\left(\frac{d^2w}{dR^2}+v\frac{dw}{dR}-\frac{vk^2}{R^2}w\right)+\frac{\Phi a}{D_0}\frac{dw}{dR} = 0, \tag{20a}$$

$$g(1)\left[\frac{d^3w}{dR^3}+\frac{d^2w}{dR^2}+\left(vk^2-2k^2-1\right)\frac{dw}{dR}+\left(3k^2-vk^2\right)w\right]+\frac{dg(1)}{dR}\left(\frac{d^2w}{dR^2}+v\frac{dw}{dR}-vk^2w\right)-\frac{Ka^3}{D_0}w = 0, \tag{20b}$$

where Φ and K are the rotational and translational spring constants, respectively.

For the conventional clamped, simply supported, free, and sliding supports, their boundary condition expressions are obtained after a proper selection of the Φ and K values and omitted here for simplicity. Four boundary/regularity conditions for a solid circular plate can be explicitly expressed through a proper combination of Eqs. (18)–(20). Then, the fourth-order governing equation (15) is solved using the GDQR.

Resembling governing equations of structural Euler beams and axisymmetrically bending shells of revolution (Wu and Liu, 1999b, 2000c), governing equation (15) is also a fourth-order differential equation, which has two given conditions at each of its boundaries. The domain $R \subset [0, 1]$ is divided into N points using Chebyshev–Gauss–Lobatto sampling points. Since two given conditions exist at both $R = 0$ and

$R = 1$, one has $n_1 = n_N = 2$. Every other point just needs to satisfy Eq. (15), i.e., $n_2 = n_3 = \cdots = n_{N-1} = 1$. When forcing these n_i in Eq. (10), the GDQR expression in this case is written as

$$\frac{\mathrm{d}^m w(R_i)}{\mathrm{d}R^m} = \sum_{j=1}^{N+2} E_{ij}^{(m)} U_j \quad (i = 1, 2, \dots N), \tag{21}$$

where $\{U\}^{\mathrm{T}} = \{U_1, U_2, \dots, U_{N+2}\} = \left\{w_1, w_1^{(1)}, w_2, w_3 \dots, w_{N-1}, w_N, w_N^{(1)}\right\}$.

Eq. (21) and its corresponding weighting coefficients have been derived and used by Wu and Liu (1999b, 2000c, 2001a) and will be applied directly in this work. Substituting Eq. (21) into Eq. (15) and Eqs. (18)–(20), respectively, the GDQR expressions are obtained for the whole problem.

$$\begin{aligned} &C_0 \sum_{j=1}^{N+2} E_{ij}^{(4)} U_j + 2\left(\frac{C_0}{R_i} + C_1\right) \sum_{j=1}^{N+2} E_{ij}^{(3)} U_j + \left(-\frac{1+2k^2}{R_i^2} C_0 + \frac{2+v}{R_i} C_1 + C_2\right) \sum_{j=1}^{N+2} E_{ij}^{(2)} U_j \\ &\quad + \left(\frac{1+2k^2}{R_i^3} C_0 - \frac{1+2k^2}{R_i^2} C_1 + \frac{v}{R_i} C_2\right) \sum_{j=1}^{N+2} E_{ij}^{(1)} U_j + \left(\frac{k^4 - 4k^2}{R_i^4} C_0 + \frac{3k^2}{R_i^3} C_1 - \frac{vk^2}{R_i^2} C_2\right) w_i \\ &\quad = \frac{\rho t_0 \omega^2 a^4}{D_0} w_i, \quad (i = 2, 3, \dots, N-1) \end{aligned} \tag{22}$$

where

$$C_0 = f^2(R_i), \quad C_1 = \frac{1}{f(R_i)} \frac{\mathrm{d}g(R_i)}{\mathrm{d}R}, \quad C_2 = \frac{1}{f(R_i)} \frac{\mathrm{d}^2 g(R_i)}{\mathrm{d}R^2}. \tag{23}$$

$$w_1 = 0, \quad \sum_{j=1}^{N+2} E_{1j}^{(2)} U_j = 0, \qquad (k = 1, 3, 5, 7, \dots) \tag{24}$$

$$w_1^{(1)} = 0, \quad \sum_{j=1}^{N+2} E_{1j}^{(3)} U_j + \frac{1}{g(0)} \frac{\mathrm{d}g(0)}{\mathrm{d}R} \sum_{j=1}^{N+2} E_{1j}^{(2)} U_j = 0, \quad (k = 0, 2, 4, 6, \dots) \tag{25}$$

$$g(1)\left(\sum_{j=1}^{N+2} E_{Nj}^{(2)} U_j + v w_N^{(1)} - vk^2 w_N\right) + \frac{\Phi a}{D_0} w_N^{(1)} = 0, \tag{26a}$$

$$\begin{aligned} &g(1)\left[\sum_{j=1}^{N+2} E_{Nj}^{(3)} U_j + \sum_{j=1}^{N+2} E_{Nj}^{(2)} U_j + \left(vk^2 - 2k^2 - 1\right) w_N^{(1)} + \left(3k^2 - vk^2\right) w_N\right] + \frac{\mathrm{d}g(1)}{\mathrm{d}R}\left(\sum_{j=1}^{N+2} E_{Nj}^{(2)} U_j \right. \\ &\quad \left. + v w_N^{(1)} - vk^2 w_N\right) - \frac{Ka^3}{D_0} w_N = 0 \end{aligned} \tag{26b}$$

The derivatives of the function $g(R_i)(i = 1, 2, \dots, N)$ can be obtained using Eq. (12), since $g(R_i)$ are known from Eqs. (13) and (14). The governing equation (22), along with its four conditions selected from a combination of Eqs. (24)–(26), is reduced to a standard eigenvalue equation of order $(N-2) \times (N-2)$. The detailed solution procedures are omitted for simplicity. The obtained results are shown in Tables 1–5 with Poisson's ratio $v = 0.3$ used in all tables of this work.

Table 1
Values of $\omega_{mk}a^2\sqrt{\rho t/D}$ for the uniform plates

Boundary conditions	m	The number of circumferential nodal lines, k					
		0	1	2	3	4	5
Clamped	0	10.216	21.260	34.877	51.030	69.666	90.739
	0[a]	10.216	21.260	34.877			
	1	39.771	60.829	84.583	111.021	140.108	171.803
	1[a]	39.771	60.829	84.584			
	2	89.104	120.079	153.815	190.304	229.519	271.428
	2[a]	89.103	120.077	153.830			
	3	158.184	199.053	242.721	289.180	338.411	390.389
	4	247.006	297.760	351.336	407.730	466.925	528.902
	5	355.569	416.203	479.675	545.983	615.114	687.051
Simply supported	0	4.935	13.898	25.613	39.957	56.842	76.203
	0[a]	4.935	13.898	25.613			
	1	29.720	48.479	70.117	94.549	121.702	151.518
	1[a]	29.720	48.479	70.117			
	2	74.156	102.772	134.298	168.675	205.851	245.778
	2[a]	74.156	102.772	134.290			
	3	138.318	176.801	218.203	262.485	309.607	359.532
	4	222.215	270.566	321.841	376.012	433.049	492.919
	5	325.849	384.069	445.216	509.268	576.203	645.992
Free	0	–	–	5.358	12.439	21.835	33.495
	0[a]	–	–	5.358			
	1	9.003	20.475	35.260	53.008	73.543	96.755
	1[a]	9.003	20.475	35.260			
	2	38.443	59.812	84.366	111.945	142.431	175.735
	2[a]	38.443	59.812	84.368			
	3	87.750	118.957	153.306	190.692	231.030	274.252
	3[a]	87.749	118.961	153.270			
	4	156.818	197.872	242.036	289.238	339.413	392.505
	5	245.634	296.540	350.534	407.562	467.573	530.521
Sliding support	0	–	3.082	8.785	16.902	27.343	40.056
	0[a]	–		8.785			
	1	14.682	28.399	44.904	64.130	86.004	110.464
	1[a]	14.682	28.399	44.904			
	2	49.218	72.859	99.361	128.677	160.754	195.539
	2[a]	49.218	72.860	99.359			
	3	103.499	137.025	173.442	212.716	254.806	299.671
	3[a]	103.500	137.009	173.564			
	4	177.521	220.923	267.231	316.419	368.456	423.308
	5	271.282	324.557	380.746	439.830	501.783	566.578

[a] Values obtained by Azimi (1988a,b).

5. Discussions and conclusions

The first frequencies for any number of circumferential nodal lines can be obtained accurately using only about eight sampling points in the GDQR. To present more frequencies, 25 points are used in all Tables 1–5. Then, the first six frequencies for any number of circumferential nodal lines converge to the third decimal places. Tables 1 and 2 show that the obtained frequencies for uniform plates are very accurate compared with the results (Azimi, 1988a,b), where the receptance method was applied.

Table 2
Values of $\omega_{mk}a^2\sqrt{\rho t/D}$ for the uniform plates for $Ka^3/D = 10$

$\Phi a/D$	m	The number of circumferential nodal lines, k					
		0	1	2	3	4	5
0.1	0	3.495	5.959	8.987	14.833	23.533	34.785
	0[a]	3.495	5.959	8.987			
	1	11.581	21.757	36.092	53.633	74.050	97.187
	1[a]	11.581	21.757	36.092			
	2	39.197	60.366	84.819	112.337	142.781	176.054
	2[a]	39.197	60.366	84.819			
	3	88.189	119.334	153.643	191.002	231.321	274.526
	4	157.152	198.178	242.323	289.510	339.674	392.757
	5	245.919	296.811	350.795	407.813	467.817	530.759
10	0	4.127	6.064	10.181	17.104	26.576	38.406
	0[a]	4.127	6.064	10.181			
	1	14.498	26.840	42.213	60.375	81.215	104.659
	1[a]	14.498	26.840	42.213			
	2	46.085	68.294	93.431	121.426	152.217	185.747
	2[a]	46.085	68.294	93.432			
	3	97.206	129.087	163.923	201.671	242.281	285.709
	4	167.682	209.270	253.835	301.342	351.752	405.026
	5	257.600	308.935	363.259	420.544	480.757	543.864
1000	0	4.238	6.090	10.576	18.058	28.154	40.656
	0[a]	4.238	6.090	10.576			
	1	15.387	28.744	45.107	64.251	86.066	110.479
	1[a]	15.387	28.744	45.107			
	2	49.377	72.928	99.370	128.637	160.669	195.413
	2[a]	49.376	72.928	99.371			
	3	103.495	136.964	173.331	212.558	254.601	299.420
	4	177.402	220.750	267.006	316.142	368.126	422.924
	5	271.05	324.267	380.398	439.422	501.314	566.047

[a] Values obtained by Azimi (1988a,b).

The authors have also tried the procedures that used one condition at the circular plate center (Gu and Wang, 1997; Gutierrez et al., 1996). The GDQR expression for the method of one central regularity condition is identical to that for the second-order dynamics problem (Wu and Liu, 1999b) and for the third-order Blasius problem (Liu and Wu, 2001c), where explicit GDQR coefficients have been obtained. It is quite interesting that the present GDQR produces almost the same and accurate results as those obtained by Gu and Wang (1997) and Gutierrez et al. (1996). This means that a fourth-order differential equation can be solved employing either three or four given conditions in a strong form. However, considering the analytic solution procedures, one has discarded some terms to avoid singularity. The test functions used here and in references (Gu and Wang, 1997; Gutierrez et al., 1996) are all polynomial functions, which have no singularity at the origin. Therefore, the regularity conditions corresponding to singularity conditions are automatically satisfied. This is the reason why the use of either three or four conditions gives the same results. However, it must be emphasized that this is only a coincidence and that nth-order differential equations should be solved using n given conditions.

Tables 3 and 4 present results for plates with linearly varying thickness. The GDQR results are in excellent agreement with the results obtained using the Ritz method (Gutierrez et al., 1996). Gutierrez et al. (1996) also employed the DQM but obtained poor or even divergent results for some cases. These authors have calculated all the same cases (Gutierrez et al., 1996) and obtained very good results using the present GDQR. The frequencies obtained by Romanelli et al. (1998) using the DQM differ by approximately 10%

Table 3
Values of $\omega_{mk}a^2\sqrt{\rho t_0/D_0}$ for the nonuniform plates for $\eta = -0.3$ (Columns 4–9: GDQR results)

Boundary conditions	m	The number of circumferential nodal lines, k						
		0 (Ritz)	0	1	2	3	4	5
Clamped	0	7.778	7.779	16.638	27.611	40.256	54.650	70.774
	1	32.463	32.462	49.852	69.225	90.346	113.317	138.133
	2		73.948	99.692	127.383	156.829	188.160	221.383
	3		132.125	166.197	202.183	239.914	279.54	321.079
	4		206.971	249.362	293.639	339.640	387.538	437.357
	5		298.477	349.184	401.752	456.019	512.181	570.264
Simply supported	0	4.116	4.116	11.194	20.539	31.639	44.538	59.203
	1	24.728	24.727	40.162	57.739	77.155	98.482	121.699
	2		62.071	85.819	111.634	139.289	168.888	200.426
	3		116.096	148.150	182.217	218.102	255.939	295.733
	4		186.785	227.146	269.475	313.594	359.662	407.693
	5		274.132	322.801	373.401	425.760	480.060	536.322
Free	0		–	–	4.380	9.786	16.788	25.380
	1	7.951	7.951	17.281	29.122	42.965	58.794	76.546
	2		32.675	50.308	70.274	92.235	116.24	142.252
	3		74.171	100.090	128.225	158.324	190.474	224.658
	4		132.353	166.561	202.905	241.170	281.478	323.826
	5		207.202	249.704	294.280	340.736	389.220	439.739
Sliding support	0		–	2.145	6.918	13.158	21.007	30.448
	1	11.942	11.942	23.152	36.519	51.689	68.725	87.599
	2		40.922	60.492	82.145	105.630	131.043	158.372
	3		86.607	114.497	144.411	176.149	209.833	245.464
	4		148.964	185.165	223.345	263.325	305.252	349.138
	5		227.983	272.492	318.944	367.167	417.332	469.457
$Ka^3/D_0 = 100$ $\Phi a/D_0 = 20$	0	6.994	6.994	13.409	19.478	25.345	31.996	40.013
	1	22.496	22.496	31.048	41.800	55.195	71.014	88.984
	2		45.269	62.865	83.292	105.895	130.587	157.280
	3		87.464	114.349	143.462	174.500	207.533	242.543
	4		147.865	183.263	220.708	259.991	301.240	344.459
	5		225.237	268.959	314.651	362.131	411.562	462.957
$Ka^3/D_0 = 10$ $\Phi a/D_0 = 2$	0	4.044	4.044	6.320	9.535	14.371	21.020	29.407
	1	12.628	12.629	21.842	33.740	47.667	63.567	81.371
	2		37.550	55.465	75.633	97.744	121.857	147.945
	3		79.679	105.851	134.169	164.404	196.655	230.913
	4		138.385	172.792	209.285	247.664	288.059	330.473
	5		213.637	256.295	300.99	347.541	396.099	446.676

from the values obtained by means of the Raleigh–Ritz and finite element methods. The reason is that domain decomposition is not used in the intermediate support, as pointed out in Section 1.

Table 5 shows the frequencies of plates with exponentially varying thickness. The constant ζ in $t = t_0 e^{\zeta R}$ varies from -1 to $+1$. When the inner part is thicker (negative ζ), the obtained frequencies are very accurate in comparison with those obtained by Singh and Saxena (1996). When the inner part is thinner (positive ζ), the accuracy of the obtained frequencies are very good for small positive ζ but are not that good for big positive ζ.

Table 4
Values of $\omega_{mk}a^2\sqrt{\rho t_0/D_0}$ for the nonuniform plates for $\eta = 0.3$ (Columns 4–9: GDQR results)

Boundary conditions	m	The number of circumferential nodal lines, k						
		0 (Ritz)	0	1	2	3	4	5
Clamped	0	12.663	12.663	25.607	41.762	61.313	84.092	110.021
	1	46.784	46.780	71.196	99.114	130.667	165.681	204.075
	2		103.411	139.226	178.695	221.928	268.751	319.081
	3		182.608	229.790	280.733	335.523	393.994	456.057
	4		284.389	342.920	405.298	471.580	541.614	615.306
	5		408.761	478.632	552.421	630.154	711.701	796.956
Simply supported	0	5.748	5.748	16.368	30.413	47.934	68.735	92.724
	1	34.564	34.562	56.364	81.880	111.154	143.972	180.228
	2		85.619	118.744	155.694	196.524	241.026	289.099
	3		159.201	203.656	252.013	304.323	360.391	420.114
	4		255.351	311.135	370.884	434.632	502.205	573.494
	5		374.084	441.195	512.324	587.486	666.527	749.333
Free	0		–	–	5.973	14.791	26.592	41.317
	1	10.134	10.132	23.533	41.183	62.736	87.872	116.439
	2		44.064	68.871	97.806	130.807	167.584	207.993
	3		100.651	136.763	177.020	221.397	269.648	321.638
	4		179.830	227.247	278.828	334.573	394.250	457.731
	5		281.597	340.327	403.248	470.343	541.423	616.355
Sliding support	0		–	3.666	10.377	20.361	33.378	49.345
	1	17.478	17.476	33.423	52.925	76.070	102.655	132.580
	2		57.195	84.574	115.682	150.603	189.130	231.167
	3		119.458	158.207	200.790	247.275	297.472	351.280
	4		204.299	254.393	308.396	366.370	428.127	493.567
	5		311.724	373.155	438.563	507.974	581.233	658.229
$Ka^3/D_0 = 100$ $\Phi a/D_0 = 20$	0	8.838	8.838	13.948	19.019	26.464	37.137	51.106
	1	22.185	22.182	34.482	51.630	72.983	98.032	126.553
	2		55.171	80.312	109.512	142.693	179.590	220.072
	3		112.816	149.431	190.035	234.647	283.057	335.150
	4		193.152	241.026	292.928	348.899	408.733	472.320
	5		295.914	355.030	418.230	485.526	556.750	631.782
$Ka^3/D_0 = 10$ $\Phi a/D_0 = 2$	0	3.685	3.685	5.596	9.488	17.326	28.651	43.115
	1	13.500	13.498	26.287	43.589	64.919	89.899	118.347
	2		46.609	71.283	100.110	133.019	169.716	210.055
	3		103.022	139.085	179.291	223.617	271.820	323.764
	4		182.139	229.532	281.082	336.796	396.441	459.889
	5		283.876	342.592	405.493	472.567	543.623	618.531

Even for the simplest free vibration or stability problems of circular plates, the governing differential equations have variable coefficients, and Bessel functions must be employed in analytic solutions. As regards circular plates with variable thickness, approximate or numerical methods are usually applied. An obvious advantage of the DQ technique is that it can easily deal with differential equations with variable coefficients. The differential equation and corresponding given conditions are directly converted to algebraic equations without extra calculations. Yeh et al. (1997) used the stepped reduction method to find general solutions for annular plates of variable thickness, and they converted a continuously varying

Table 5
Values of $\omega_{mk}a^2\sqrt{\rho t_0/D_0}$ for the plates with exponential thickness variation (Columns 4–9: GDQR results)

Boundary conditions	m	The number of circumferential nodal lines, k						
		0 (Ritz)	0	1	2	3	4	5
For the case of $\zeta = -1.0$								
Clamped	0	4.764	4.765	9.517	17.063	24.867	33.512	43.049
	1	22.051	22.052	33.261	46.119	59.451	73.664	88.813
	2	51.129	51.127	68.311	86.780	105.680	125.428	146.102
	3	91.995	91.986	115.072	139.206	163.701	188.998	215.194
	4		144.607	173.567	203.399	233.517	264.386	296.119
Simply supported	0	2.845	2.846	6.255	12.767	19.528	27.168	35.728
	1	17.224	17.224	27.003	38.634	50.802	63.898	77.966
	2	43.357	43.354	59.105	76.275	93.940	112.503	132.028
	3	81.288	81.271	102.921	125.719	148.937	173.003	198.002
	4		130.954	158.474	186.947	215.76	245.365	275.868
Free	0		6.204	11.544	2.030	5.503	9.556	14.406
	1		23.374	34.725	19.348	27.796	37.253	47.729
	2		52.415	69.696	47.904	61.712	76.516	92.352
	3		93.255	116.416	88.406	107.669	127.879	149.097
	4		145.866	174.886	140.744	165.534	191.211	217.860
Sliding support	0		8.320	14.970	3.704	7.562	12.128	17.502
	1		28.491	41.249	23.869	33.266	43.586	54.862
	2		60.500	79.195	55.613	70.497	86.296	103.065
	3		104.293	128.874	99.159	119.580	140.872	163.114
	4		159.849	190.295	154.497	180.498	207.317	235.052
For the case of $\zeta = 1.0$								
Clamped	0	23.239	23.235	41.583	66.357	98.035	136.183	180.631
	1	72.505	72.457	107.422	149.303	198.504	254.524	317.176
	2	153.45	152.914	204.22	262.776	328.942	402.157	482.215
	3	271.18	264.980	332.503	407.547	490.422	580.536	677.653
	4		408.784	492.454	583.877	683.307	790.143	904.118
Simply supported	0	9.007	9.005	23.948	46.330	75.697	111.558	153.734
	1	51.331	51.290	82.574	121.293	167.580	220.827	280.801
	2	124.63	123.985	171.528	226.692	289.691	359.890	437.039
	3	236.80	228.178	291.888	363.412	442.971	529.910	623.962
	4		364.069	443.894	531.715	627.726	731.274	842.069
Free	0		14.611	33.075	60.716	14.731	38.615	67.093
	1		62.540	98.020	141.853	96.084	138.448	187.466
	2		142.512	194.358	254.524	193.938	253.339	319.769
	3		254.399	322.307	398.590	322.925	398.962	482.242
	4		398.004	482.091	574.683	483.472	576.008	675.935
Sliding support	0		28.866	51.212	10.247	28.239	52.049	81.809
	1		85.345	124.401	80.681	117.514	161.229	211.565
	2		173.397	228.865	170.845	225.031	286.273	354.378
	3		293.276	364.894	291.962	362.884	441.132	526.416
	4		444.885	532.701	444.339	531.958	626.971	729.159

thickness into a stepped one. The derivation process was very complicated. The techniques used in this work directly solve the governing equations of plates with a continuously varying thickness.

This work deals with the free vibration of solid circular plates with radially variable thickness and elastic constraints using the GDQR. Two regularity conditions at the solid circular plate origin are discussed in detail. Errors in the literature are pointed out in the expression of regularity conditions and in the application of differential quadrature techniques. Numerical results are presented for a number of examples, illustrating the versatility and accuracy of the approach.

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